Direct Data-Driven Design of LPV Controllers and Polytopic Invariant Sets with Cross-Covariance Noise Bounds

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Abstract— **We propose a direct data-driven method for the concurrent computation of polytopic *robust control invariant* (RCI) sets and associated invariance-inducing control laws for *linear parameter-varying* (LPV) system. We present a data-based covariance parameterization of the gain-scheduled controller and the closed-loop dynamics, utilizing a persistently exciting state-input-scheduling trajectory gathered from the LPV system. This parameterization, along with the assumption of bounded cross-covariance noise, allows us to express the invariance condition as a set of data-based LMIs, such that the number of decision variables is *independent* of the length of the dataset. These LMIs are combined with state-input constraints framed as simple affine inequalities in a convex semi-definite program with an objective to maximize the volume of the RCI set. A numerical example demonstrates the computational effectiveness of the proposed method in synthesizing RCI sets even with large datasets.**

I. INTRODUCTION

A subset of the state-space is termed as a *robust control invariant* (RCI) set if, for a feasible controller, the states of the system initialized within the set remain in that set *ad infinitum*, for all bounded disturbances acting on the system [5]. Characterizing an RCI set and invariance-inducing controllers for a constrained dynamical system is crucial for its stability analysis and for guaranteeing safety [6]. For *linear parameter-varying* (LPV) systems, several *model-based* approaches have been proposed in the literature to compute RCI sets and associated controllers, see *e.g.*, [11], [12], [19], [23]. These methods assume the knowledge of an accurate LPV model of the system, which is anything but trivial to derive either from first principles or from gathered data [24].

To overcome the challenges of model-based control design approaches, recent contributions have advocated *direct* data-driven control paradigm [2], [7], [8], [20], synthesizing controllers directly from the data, without requiring a model of the system and obviating the need of an intermediate system identification step. In the context of invariant set computation problem, direct data-driven algorithms to synthesize RCI sets and invariance-inducing controllers have been presented in [1], [4], [14], [27] for *linear time-invariant* (LTI) systems, while for LPV systems, to the best of our knowledge, only recent works [15], [16] have addressed concurrent RCI set and controller synthesis. The methods in [15], [16] follow the *set-membership* paradigm [17] or a resembling *data informativity* [25] framework: an invariance-inducing controller is sought for *all* consistent LPV matrices compatible with the data and satisfying a given noise bound, rather than a singleton. Enforcing invariance condition robustly *for all* matrices in the feasible model set, renders the obtained set invariant for the true system as well. Note that, this result comes at a price of conservatism stemming from the lack of model knowledge, which, nonetheless can be reduced by increasing the number of gathered data samples. The reason being that the feasible model set 'shrinks' as the number of data samples increase. However, the procedures to enforce invariance proposed in [15], [16] introduce decision variables whose size grows with the length of the dataset, making them inefficient and potentially prohibitive for large datasets.

To overcome this issue, we present an alternative approach based on a data-driven covariance parameterization of the controller along with an additional assumption of bounded cross-covariance noise. We show that under these parameterization and assumption, RCI sets can be computed with a significantly lower computational expense for large datasets. To this end, we built upon the method presented in [15], namely, the set invariance is guaranteed robustly for all scheduling parameters and disturbances in given sets using a gain-scheduled state-feedback controller, by employing a full block S-procedure. However, unlike the data-based parameterization of the feasible model set considered in [15], we devise a parameterization of the controller which consequently characterizes a data-based representation of the *closed-loop* LPV matrices. This allows us to focus directly on the *closed-loop dynamics* over which we wish to have safety guarantees. To achieve this, we suitably adapt and extend the *covariance policy* parameterization recently introduced in [9] for LTI systems, to gain-scheduled controllers in the LPV setting. This parameterization choice results in a number of decision variables that do not increase with the length of the dataset, thereby overcoming the limitation of the data-based closed-loop LPV representation adopted in [26].

Furthermore, by assuming cross-covariance noise bounds (similar to those considered in [22] for LTI systems) between the disturbance vector and a variable constructed from state-input-scheduling vectors, we show that data-based invariance conditions can be formulated such that the number of decision variables in the LMIs remains constant, regardless of the dataset length. As the cross-covariance noise bounds are in general not known *a priori*, we also present a method to approximate such bounds from the available information.

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The paper is organized as follows: The problem addressed in this work is formalized in Section II. The proposed data-based controller parameterization and closed-loop dynamics are introduced in Section III. In Sections IV, we present tractable, data-driven reformulations of the state/input constraints. In Section V, we introduce cross-covariance noise bounds and present a method for their estimation. Our main result is presented in Section VI, where we derive data-based LMI conditions to compute the RCI sets and controllers. The effectiveness of the proposed algorithm is showcased via a numerical example in Section VII. The paper concludes with some remarks and directions for future work.

Notations and Preliminaries

Let $\mathbb{N}_m^n := \{m, \ldots, n\}$ denote a set of natural numbers between two integers m and n, $m \leq n$. An identity matrix of size $n \times n$ is denoted by I_n and $\mathbf{1}_n$ denotes the n-dimensional vector of all-ones. We denote by $P \succ 0 (\succeq 0)$ a positive (semi) definite matrix P. The $\star$'s within an LMI denote the symmetric matrix entries $\left[\begin{smallmatrix} A & B \\ B^\top & C \end{smallmatrix}\right] := \left[\begin{smallmatrix} A & B \\ \star & C \end{smallmatrix}\right]$. For two vectors $a, b \in \mathbb{R}^n$, $a \leq b$ denotes element-wise inequality. Given a matrix $A \in \mathbb{R}^{m \times n}$, $\overrightarrow{A} := \left[\begin{smallmatrix} a_1^\top & \cdots & a_n^\top \end{smallmatrix}\right]^\top \in \mathbb{R}^{mn}$ denote its vectorization, stacking the columns a_j, $j \in \mathbb{N}_1^n$. The convex-hull of a finite set $\Theta := \{\theta^j \in \mathbb{R}^n, j \in \mathbb{N}_1^r\}$, is denoted by, $\mathsf{Conv}(\Theta) := \left\{\theta \in \mathbb{R}^n : \theta = \sum_{j=1}^r \alpha_j \theta^j, \text{s.t } \sum_{j=1}^r \alpha_j = 1, \alpha_j \in [0,1]\right\}$. The Kronecker product between matrices A and B is denoted by $A \otimes B$. All unknown matrix variables to be computed are written in a boldface font, *e.g.*, $\mathbf{W}, \mathbf{K}$ etc.

The following results will be used in the paper:

Lemma 1 (Vectorization): For matrices $A \in \mathbb{R}^{k \times l}$, $B \in \mathbb{R}^{l \times m}$, $C \in \mathbb{R}^{m \times n}$ and $D \in \mathbb{R}^{k \times n}$, the matrix equation $ABC = D$ is equivalent to,

$$(C^\top \otimes A)\overrightarrow{B} = \overrightarrow{ABC} = \overrightarrow{D}$$

Lemma 2 (Quadratic Constraint for Polytopes [10]): A symmetric polytopic set $\mathcal{X} := \{x \in \mathbb{R}^n : -h \leq Hx \leq h\}$ with $h \in \mathbb{R}_+^{n_x}$, satisfies

$$\mathcal{X} := \bigcap_{\mathbf{\Gamma} \in \mathcal{G}} \{x \in \mathbb{R}^n : (h + Hx)^\top \mathbf{\Gamma}(h - Hx) \geq 0\},$$

where $\mathcal{G} := \{\mathbf{\Gamma} : \mathbf{\Gamma} \in \mathbb{D}_+^{n_x}\}$ is the set of diagonal matrices with positive entries.

II. PROBLEM FORMULATION

Let us consider the following LPV A-affine (LPV-A [3]) structure with a constant input matrix as the data-generating system:

$$x_{k+1} = \mathcal{A}(p_k)x_k + B_o u_k + w_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control input, $p_k \in \mathbb{R}^s$ is the scheduling parameter and $w_k \in \mathbb{R}^n$ is the disturbance vector, at time k. The matrix function $\mathcal{A}(p_k)$ depends linearly [1] on the scheduling parameter vector p_k,

$$\mathcal{A}(p_k) = \sum_{j=1}^s p_{k,j} A_o^j, \tag{2}$$

where $p_{k,j}$ denotes the j-th element of $p_k \in \mathbb{R}^s$.

The LPV system (1) can be compactly written as,

$$x_{k+1} = \underbrace{\begin{bmatrix} B_o & A_o^1 & \cdots & A_o^s \end{bmatrix}}_{M_o} \begin{bmatrix} u_k \\ p_k \otimes x_k \end{bmatrix} + w_k. \tag{3}$$

The system is subjected to the following polytopic constraints on the state, input, scheduling and disturbance signals, respectively:

$$\mathcal{X} := \{x : H_x x \leq h_x\}, \; \mathcal{U} := \{u : H_u u \leq h_u\}, \tag{4a}$$
$$\mathcal{P} := \mathsf{Conv}(\{p^j\}, j \in \mathbb{N}_1^{v_p}) \tag{4b}$$
$$\mathcal{W} := \{w : -\mathbf{1}_{n_w} \leq H_w w \leq \mathbf{1}_{n_w}\}, \tag{4c}$$

where $h_x \in \mathbb{R}^{n_x}, h_u \in \mathbb{R}^{n_u}$ are given vectors, H_x, H_u, H_w, are given matrices of compatible dimensions and $\{p^j\}, j \in \mathbb{N}_1^{v_p}$, are given vertices of the scheduling parameter set $\mathcal{P}$. Note that the states are affected by *unknown* disturbance samples w_k, which are assumed to belong to a bounded *known* disturbance set $w_k \in \mathcal{W}$ for all $k \in \mathbb{N}$.

With these constraints, we aim to design an invaraince-inducing scheduling parameter-dependent state-feedback control law given by,

$$u_k = \mathcal{K}(p_k)x_k, \tag{5}$$

where the function $\mathcal{K}(p_k)$ is linearly dependent on p_k, *i.e.*,

$$\mathcal{K}(p_k) = \sum_{j=1}^s p_{k,j}\mathbf{K}^j,$$

and $\mathbf{K}^j \in \mathbb{R}^{m \times n}$, $j \in \mathbb{N}_1^s$ are feedback-gain matrices. By defining $\mathbf{K} := \begin{bmatrix} \mathbf{K}^1 & \cdots & \mathbf{K}^s \end{bmatrix} \in \mathbb{R}^{m \times ns}$, the control law can be expressed as,

$$u_k = \mathcal{K}(p_k)x_k = \mathbf{K}(p_k \otimes x_k). \tag{6}$$

By substituting (6) in (3), the resulting closed-loop dynamics is given by,

$$x^+ = M_o \begin{bmatrix} \mathbf{K} \\ I_{ns} \end{bmatrix} (p \otimes x) + w, \tag{7}$$

where the successor state x_{k+1} is denoted as x^+, and the time dependence of the signals is dropped for brevity.

Let us now consider the following polytopic set,

$$\mathcal{S} := \left\{x \in \mathbb{R}^n : -\mathbf{1}_{n_c} \leq C\mathbf{W}^{-1}x \leq \mathbf{1}_{n_c}\right\}, \tag{8}$$

where $C \in \mathbb{R}^{n_c \times n}$ is a user-defined fixed matrix which determines the representational complexity of the set, with $2n_c$ being the number of hyperplanes, and $\mathbf{W} \in \mathbb{R}^{n \times n}$ is the matrix to be computed such that the set $\mathcal{S}$ is an RCI set according to the following definition:

[1] it can be considered *affine* in the parameter by adjoining a new constant $p_{k,0} = 1$ to the scheduling vector p.

Definition 1 (Robust invariance): A set $\mathcal{S} \subseteq \mathbb{R}^n$ is *robustly invariant* for the system (7), if for any given $p \in \mathcal{P}$, there exists a controller gain matrix $\mathbf{K}$ such that

$$x \in \mathcal{S} \Rightarrow x^+ \in \mathcal{S}, \ \forall w \in \mathcal{W}. \tag{9}$$

Moreover, the RCI set has to satisfy the state and input constraints given in (4), $\mathcal{S} \subseteq \mathcal{X}$ and $\mathcal{K}(p)\mathcal{S} \subseteq \mathcal{U}$, *i.e.*,

$$x \in \mathcal{S} \Rightarrow x \in \mathcal{X}, \tag{10}$$

$$x \in \mathcal{S} \Rightarrow u = \mathbf{K}(p \otimes x) \in \mathcal{U} \quad \forall p \in \mathcal{P}. \tag{11}$$

In this work, we assume that the system matrices $A_o^j \in \mathbb{R}^{n\times n}, j \in \mathbb{N}_1^s$ and B_o in (3) are *unknown*, instead, we assume that a dataset $\{x_k, u_k, p_k\}_{k=1}^{T+1}$ consisting of $T+1$ state-input-scheduling samples generated from system (3) is available.

We define the following data matrices,

$$X^+ := [x_2 \quad x_3 \quad \cdots \quad x_{T+1}] \in \mathbb{R}^{n\times T}, \tag{12a}$$

$$U_0 := [u_1 \quad u_2 \quad \cdots \quad u_T] \in \mathbb{R}^{m\times T}, \tag{12b}$$

$$X_0 := [p_1 \otimes x_1 \quad \ldots \quad p_T \otimes x_T] \in \mathbb{R}^{ns\times T}, \tag{12c}$$

and let us assume that the gathered data satisfies the following condition:

Assumption 1 (Persistence of excitation): The matrix

$$\Phi := \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} \in \mathbb{R}^{(m+ns)\times T} \tag{13}$$

has a full row rank, *i.e.*, $\text{rank}(\Phi) = m + ns$.

With this setting, we now formalize the direct data-driven invariant set computation problem addressed in this paper.

Problem 1: Given data matrices (X^+, U_0, X_0) defined in (12), the constraints sets (4) and a fixed matrix C in (8), compute the matrix $\mathbf{W}$ defining the set $\mathcal{S}$ in (8) and feedback controller gains $\mathbf{K}$ in (6) such that: (i) the set $\mathcal{S}$ is robustly invariant for the closed-loop dynamics, satisfying condition (9); (ii) the state and input constraints (10) and (11) are satisfied; (iii) $\mathcal{S}$ has the largest possible volume.

III. Data-driven parameterization of the controller and closed-loop dynamics

In this section, we present a data-based parameterization of the gain-scheduled controller in (6) and the associated closed-loop LPV matrices. To this end, we adapt the sample covariance policy parameterization introduced in [9] for the LTI systems and extend it to our LPV setting.

Let $\Sigma_p \in \mathbb{R}^{(m+ns)\times(m+ns)}$ be the sample covariance matrix defined as follows,

$$\Sigma_p := \frac{1}{T}\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}^\top = \frac{1}{T}\Phi\Phi^\top, \tag{14}$$

which we use to parameterize any given controller gain matrix $\mathbf{K}$ in (6) as,

$$\begin{bmatrix} \mathbf{K} \\ I_{ns} \end{bmatrix} = \Sigma_p \mathbf{G}, \tag{15}$$

where $\mathbf{G} \in \mathbb{R}^{(m+ns)\times ns}$ is the matrix variable to be computed, whose dimensions are *independent* of the number of data samples T. Thus, this parameterization is computationally efficient for large datasets. Note that under Assumption 1, the covariance Σ_p in (14) is positive definite and there exists a unique matrix $\mathbf{G}$ satisfying (15).

As the data-matrices in (12) satisfy the system dynamics (3), we have,

$$X^+ = M_o\Phi + W_0, \tag{16}$$

where, $W_0 := [w_1, \cdots, w_T]$ is an *unknown* realization of the disturbance signal belonging to the set $W_0 \in \mathcal{W}_T \subset \mathbb{R}^{n\times T}$ defined as follows,

$$\mathcal{W}_T := \{W : -\bar{\mathbf{1}} \leq H_w W \leq \bar{\mathbf{1}}\}, \tag{17}$$

with $\bar{\mathbf{1}} := [\,\mathbf{1}_{n_w} \ \mathbf{1}_{n_w} \cdots \ \mathbf{1}_{n_w}\,] \in \mathbb{R}^{n_w\times T}$.

Let us now define the following matrices,

$$U_p := \frac{1}{T}U_0\Phi^\top, \qquad X_p := \frac{1}{T}X_0\Phi^\top, \tag{18a}$$

$$X_p^+ := \frac{1}{T}X^+\Phi^\top, \qquad W_p := \frac{1}{T}W_0\Phi^\top. \tag{18b}$$

From (16), the matrices in (18) satisfy the following dynamics,

$$X_p^+ = M_o\Sigma_p + W_p, \tag{19}$$

where we note the matrix W_p is unknown, constructed from an unknown true disturbance realization sequence $W_0 \in \mathcal{W}_T$.

It follows that for a given feedback gain matrix $\mathbf{K}$, the closed-loop matrices of the true LPV system are given as,

$$M_o\begin{bmatrix} \mathbf{K} \\ I_{ns} \end{bmatrix} = M_o\Sigma_p\mathbf{G} = (X_p^+ - W_p)\mathbf{G}, \tag{20}$$

with the following consistency conditions on $\mathbf{G}$ obtained from (14) and (15),

$$\mathbf{K} = U_p\mathbf{G}, \tag{21a}$$

$$I_{ns} = X_p\mathbf{G}. \tag{21b}$$

Based on (20), the closed-loop dynamics (7) can be expressed in terms of data, disturbance matrix W_p, and the design parameter $\mathbf{G}$ as follows,

$$x^+ = (X_p^+ - W_p)\mathbf{G}\,(p \otimes x) + w. \tag{22}$$

IV. Convex formulations of state, input, and consistency constraints

In this section, we present a coordinate transformation, such that in the new coordinate space the candidate RCI set is known. We enforce robust invariance of this set in the new space. This allows us to express the state, input, consistency constraints as well as the invariance condition as tractable convex constraints.

Let us consider the following state transformation [11],

$$\theta = \mathbf{W}^{-1}x \Leftrightarrow x = \mathbf{W}\theta, \tag{23}$$

and, the set $\mathcal{S}$ in (8) in the transformed θ-state-space is given as,

$$\mathcal{S} := \{\mathbf{W}\theta \in \mathbb{R}^n : \theta \in \Theta\}, \tag{24}$$

where Θ is a symmetric set defined as follows:

$$\Theta \triangleq \{\theta \in \mathbb{R}^n : -\mathbf{1}_{n_c} \leq C\theta \leq \mathbf{1}_{n_c}\}. \tag{25}$$

Note that the candidate invariant set Θ in the θ-state-space, is a *known* symmetric set, as C is a known user-defined matrix. The set Θ can be expressed as a convex hull of its *known* vertices $\left\{\theta^1, \ldots, \theta^{2v_\theta}\right\}$:

$$\Theta = \mathsf{Conv}\left(\left\{\theta^i\right\},\ i \in \mathbb{N}_1^{2v_\theta}\right), \tag{26}$$

where $v_\theta > 0$ is determined by the choice of C. The corresponding set $\mathcal{S}$ in the x-state-space will be completely determined by $\mathbf{W}$, which we aim to compute.

A. System constraints

We now express the state and input constraints (4) in the θ-state-space with the transformation considered in (23). Satisfaction of these inequalities constraints at the vertices $\{\theta^i\}_{i=1}^{2v_\theta}$ ensures that they are satisfied over the whole set Θ as well, due to convexity. We can write the state constraints (10) in terms of $\mathbf{W}$ in the transformed space as follows:

$$H_x\mathbf{W}\theta \leq h_x, \forall\theta \in \Theta \ \Leftrightarrow\ H_x\mathbf{W}\theta^i \leq h_x,\ i \in \mathbb{N}_1^{2v_\theta} \tag{27}$$

Similarly, the control input constraints in (11) are given by

$$H_u\mathbf{K}(p_k \otimes \mathbf{W}\theta) \leq h_u \Leftrightarrow H_uU_p\mathbf{G}(p_k \otimes \mathbf{W}\theta) \leq h_u, \tag{28}$$

where we have substituted the controller gain $\mathbf{K}$ in terms of the data matrix U_p as given in (21a). In order to resolve the bilinearity between $\mathbf{G}$ and $\mathbf{W}$ in (28), let the matrix $\mathbf{G}$ defined in (15) be written as $\mathbf{G} = [\mathbf{G}^1, \cdots, \mathbf{G}^s]$ and let us define new matrix variables $\mathbf{N}^l \in \mathbb{R}^{(m+ns)\times n}$ for $l \in \mathbb{I}_1^s$ as follows:

$$\mathbf{N} := \begin{bmatrix}\mathbf{N}^1 & \cdots & \mathbf{N}^s\end{bmatrix} = \begin{bmatrix}\mathbf{G}^1\mathbf{W} & \cdots & \mathbf{G}^s\mathbf{W}\end{bmatrix}, \tag{29}$$

with $\mathbf{G}^l := \mathbf{N}^l\mathbf{W}^{-1}$. The term on the left hand side of the inequality in (28) can be expressed as,

$$\begin{aligned}\mathbf{G}(p_k \otimes \mathbf{W}\theta) &= \left[\mathbf{G}^1\mathbf{W}p_{k,1} + \cdots + \mathbf{G}^s\mathbf{W}p_{k,s}\right]\theta \\ &= \underbrace{\begin{bmatrix}\mathbf{G}^1\mathbf{W} & \cdots & \mathbf{G}^s\mathbf{W}\end{bmatrix}}_{\mathbf{N}}(p_k \otimes \theta).\end{aligned} \tag{30}$$

and thus, the input constraints in (28) can be re-written as,

$$H_uU_p\mathbf{N}(p \otimes \theta) \leq h_u, \forall(\theta, p) \in (\Theta, \mathcal{P}) \ \Leftrightarrow \tag{31}$$

$$H_uU_p\mathbf{N}(p^j \otimes \theta^i) \leq h_u,\ i \in \mathbb{N}_1^{2v_\theta}, j \in \mathbb{N}_1^{v_p}. \tag{32}$$

Note that (31) and (32) are equivalent as the Kronecker product map is linear in each of its arguments (p, θ) and the sets $\mathcal{P}, \Theta$ and $\mathcal{U}$ are convex.

B. Consistency constraints

Let us now express the consistency condition in (21b) in terms of the introduced matrix variable $\mathbf{N}$. Condition (21b) can be re-written as

$$I_{ns} = [X_p\mathbf{N}^1\mathbf{W}^{-1} \quad \cdots \quad X_p\mathbf{N}^s\mathbf{W}^{-1}],$$

leading to the following linear equality constraints,

$$\mathbb{I}^l\mathbf{W} = X_p\mathbf{N}^l,\ l \in \mathbb{N}_1^s, \tag{33}$$

where $\mathbb{I}^l \in \mathbb{R}^{ns\times n}$ denotes the matrix constructed from the corresponding columns of the identity matrix I_{ns}.

C. Invariance conditions

We now express the system dynamics in the θ-state-space and present conditions for the set Θ in (26) to be invariant. Using (23) and (30), the closed-loop dynamics (22) can be written as

$$\begin{aligned}x^+ = \mathbf{W}\theta^+ &= (X_p^+ - W_p)\mathbf{G}(p \otimes \mathbf{W}\theta) + w, \\ &= (X_p^+ - W_p)\mathbf{N}(p \otimes \theta) + w.\end{aligned} \tag{34}$$

Considering the closed-loop dynamics (34), we now state two equivalent invariance conditions for the set Θ.

Lemma 3: If the set Θ in (25) is robustly invariant for system (34) then the following two statements are equivalent:

(i) for all $\theta \in \Theta$, for any given $p \in \mathcal{P}$, $\forall w \in \mathcal{W}$,

$$\theta^+ = \left(\mathbf{W}^{-1}(X_p^+ - W_p)\mathbf{N}(p \otimes \theta) + \mathbf{W}^{-1}w\right) \in \Theta. \tag{35}$$

(ii) for each vertex $\theta^i,\ i \in \mathbb{N}_1^{2v_\theta}$ of the set Θ, and for each vertex $p^j,\ j \in \mathbb{N}_1^{v_p}$ of the set $\mathcal{P}$, $\forall w \in \mathcal{W}$,

$$\theta^{i,j+} := \left(\mathbf{W}^{-1}(X_p^+ - W_p)\mathbf{N}(p^j \otimes \theta^i) + \mathbf{W}^{-1}w\right) \in \Theta. \tag{36}$$

Proof: See Appendix IX. ■

The second condition (36) in Lemma 3 allows us to enforce robust invariance of the set Θ, at a finite set of known vertices of $\mathcal{P}$ and Θ, thus, we will use condition (36) to compute the RCI set and controller parameters.

V. Cross-covariance noise bounds

Note that the dynamics (34) and in turn, condition (36) is parameterized in terms of matrix W_p which is *unknown*, as it depends on a true realization W_0 of the disturbance sequence as defined in (18). To solve Problem 1 (point i), the invariance condition (36) should be satisfied (conservatively) for *all* possible realizations of the disturbance sequences $W_0 \in \mathcal{W}_T$, in turn, for all W_p in some set $\mathcal{W}_p$. To this end, let us now characterize a set $\mathcal{W}_p$, by utilizing the known disturbance bounds $\mathcal{W}_T$ and available data. This set will be later used to enforce (36) robustly *for all* $W_p \in \mathcal{W}_p$.

We assume the following bounds on the sample cross-covariance of the noise w_k and a variable $r_k := [u_k^\top\ (p_k \otimes x_k)^\top]^\top \in \mathbb{R}^{m+ns}$,

$$-c_j \leq \frac{1}{T}\sum_{k=1}^{T} w_k r_{k,j} \leq c_j, \quad j \in \mathbb{N}_1^{m+ns}, \tag{37}$$

where $c_j \in \mathbb{R}_+^n$ are given vectors and $r_{k,j}$ denotes the j-th element of r_k. The bounds in (37) define polyhedral bounds on the sample cross-covariance of the disturbance w_k and an instrumental variable r_k.

Then, from the definition of W_p in (18), bounds in (37) can be expressed as,

$$\mathcal{W}_p := \left\{\overrightarrow{W_p} \in \mathbb{R}^{n(m+ns)} : -w^{\max} \leq \overrightarrow{W_p} \leq w^{\max}\right\}, \tag{38}$$

where $w^{\max} := [c_1^\top \ \cdots \ c_{m+ns}^\top]^\top$ and $\overrightarrow{W_p}$ denotes the vectorization of W_p.

Remark 1: For LTI systems, similar bounds were introduced in [13] for parameter bounding identification and analysed in [22] to provide informativity conditions for data-driven control. The choice of the instrumental variable r and estimating the bounds (37) from data are discussed in [13]. It is suggested to choose an instrument r_k that is correlated with the inputs, but uncorrelated with the noise w_k. The variable $r_k := [u_k^\top \ (p_k \otimes x_k)^\top]^\top$ satisfies these requirements.

The characterization of $\mathcal{W}_p$ in (38) depends on the cross-covariance bounds $c_j \in \mathbb{R}_+^n$ which are typically not available in practice. For LTI case, strategies to estimate such bounds from data are discussed in [13], under some assumptions on the noise statistics. Instead, here we propose an alternative approach that relies on the *known* bounds on the process noise (see (17)), to estimate the cross-covariance bounds and to approximate the set $\mathcal{W}_p$ in (38).

To this end, we recall that the true disturbance realization belongs to the set $\overrightarrow{W}_0 \in \mathcal{W}_T$ where using Lemma 1, $\mathcal{W}_T$ in (17) can be expressed as,

$$\mathcal{W}_T := \left\{ \overrightarrow{W} : -\mathbf{1}_{Tn_w} \leq \bar{H}_w \overrightarrow{W} \leq \mathbf{1}_{Tn_w} \right\}, \tag{39}$$

with $\bar{H}_w := (I_T \otimes H_w)$. By noticing that $\overrightarrow{W_p} := \frac{1}{T}(\Phi \otimes \overrightarrow{W_0})$, the bounds $w^{\max}, w^{\min}$ can be computed by solving the following *linear programs* (LP):

$$e_j^\top w^{\max} := \max_{\overrightarrow{W} \in \mathcal{W}_T} \frac{1}{T} e_j^\top (\Phi \otimes I_n) \overrightarrow{W} \tag{40a}$$

$$e_j^\top w^{\min} := \min_{\overrightarrow{W} \in \mathcal{W}_T} \frac{1}{T} e_j^\top (\Phi \otimes I_n) \overrightarrow{W} \tag{40b}$$

for all $j \in \mathbb{N}_1^{n(m+ns)}$, where e_j is the j-th column vector of the identity matrix $I_{n(m+ns)}$, $w^{\max} \in \mathbb{R}_+^{n(m+ns)}$. As the constraint set $\mathcal{W}_T$ is symmetric $w^{\min} = -w^{\max}$ and only one of the LPs in (40) is required to be solved.

The invariance condition in (36) can be enforced for all $\overrightarrow{W}_p$ in $\mathcal{W}_p$ defined by (38) and (40) via the S-procedure. In particular, we will employ Lemma 2 to express (38) in terms of quadratic constraints as

$$(w^{\max} + \overrightarrow{W_p})^\top \mathbf{\Lambda} (w^{\max} - \overrightarrow{W_p}) \geq 0 \tag{41}$$

with the matrix variable $\mathbf{\Lambda} \in \mathbb{D}_+^{n(m+ns)}$ to be computed.

Remark 2 (Efficiency vs *conservativeness):* Note that as the number of variables $\mathbf{\Lambda}$ are *independent* of the length of the data T, the proposed approach has a constant computational complexity $\mathcal{O}(1)$ w.r.t. T. Therefore, it is computationally efficient even for large datasets. This computational advantage comes at the price of conservativeness, as the set $\mathcal{W}_p$ is an outer-approximation of the true set $\mathcal{W}_p^o$ to which $\overrightarrow{W}_p$ belongs to, *i.e.*, $\mathcal{W}_p^o \subseteq \mathcal{W}_p$. Characterizing tighter approximations of the true set $\mathcal{W}_p^o$ will be investigated in future works.

Remark 3 (Comparison with [15]): In [15], an invariance inducing controller is sought for all feasible model matrices consistent with *open-loop* data and known noise bounds. A potential limitation of this approach is that the S-procedure to enforce invariance for all model matrices introduces number of decision variables which grows with the dataset length T. This has been remedied in the present work by employing a data-based covariance control parameterization and assuming cross-covariance noise bounds. Unlike the open-loop representation considered in [15], the present work is focused on data-based *closed-loop* parameterization, in particular, we remark that for some $\mathbf{G}$ satisfying (15), the set $\mathcal{M}_\mathbf{G} := \left\{ M_\mathbf{G} : M_\mathbf{G} := (X_p^+ - W_p)\mathbf{G}, I_{ns} = X_p \mathbf{G} \right\}$ represents the *superset* of a set of all *closed-loop* LPV matrices compatible with the data. The unknown matrix W_p in the closed-loop dynamics introduces conservativeness, nonetheless, we have shown that W_p is bounded in a set $\mathcal{W}_p$ which can be characterized using known disturbance bounds and the number of optimization variables required to enforce invariance $\forall W_p \in \mathcal{W}_p$ does not grow with T.

VI. Data-based LMI condition for invariance

With these considerations, we now state and prove a data-based sufficient condition to render the set Θ invariant with an associated LPV controller parameterized as in (15).

Using the vectorization Lemma 1, we rewrite the closed-loop dynamics (36) at the vertices θ^i, p^j as follows:

$$\mathbf{W}\theta^{i,j+} = \underbrace{X_p^+ \mathbf{N}(p^j \otimes \theta^i)}_{:=\mathcal{G}_{ij}(\mathbf{N})} - \underbrace{\left(\left(\mathbf{N}(p^j \otimes \theta^i) \right)^\top \otimes I_n \right)}_{:=\mathcal{F}_{ij}(\mathbf{N})} \overrightarrow{W_p} + w, \tag{42}$$

with $i \in \mathbb{N}_1^{2v_\theta}$ and $j \in \mathbb{N}_1^{v_p}$. Let us introduce new matrix variables $\mathbf{V}_{ijk} \in \mathbb{R}^{n \times n}$ and signals $\xi_{ijk} := \mathbf{V}_{ijk}^{-1} \mathbf{W} \theta^{i,j+}$, for $k \in \mathbb{N}_1^{n_c}$, $i \in \mathbb{N}_1^{2v_\theta}$, $j \in \mathbb{N}_1^{v_p}$. With theses, the dynamics (42) can be written as,

$$\mathcal{G}_{ij}(\mathbf{N}) - \mathcal{F}_{ij}(\mathbf{N}) \overrightarrow{W_p} + w - \mathbf{V}_{ijk}\xi_{ijk} = 0. \tag{43}$$

We now present the data-based sufficient LMI conditions for the invariance in the following Theorem.

Theorem 4 (Data-based LMI conditions for invariance): Given a fixed matrix $C \in \mathbb{R}^{n_c \times n}$, if there exists $\mathbf{W} \in \mathbb{R}^{n \times n}$, $\mathbf{N} \in \mathbb{R}^{(m+ns) \times ns}$, and variables $\{\boldsymbol{\phi}_{ijk} \in \mathbb{R}_+, \mathbf{\Gamma}_{ijk} \in \mathbb{D}_+^{n_w}, \mathbf{\Lambda}_{ijk} \in \mathbb{D}_+^{n(m+ns)}, \boldsymbol{X}_{ijk}, \mathbf{V}_{ijk} \in \mathbb{R}^{n \times n}\}$ that satisfy the following constraints and LMIs

$$\mathbb{I}_{ns \times n}^l \mathbf{W} = X_p \mathbf{N}^l, \quad \forall l \in \mathbb{N}_1^s, \tag{44}$$

$$\begin{bmatrix} \mathbf{W}^\top + \mathbf{W} - \boldsymbol{X}_{ijk} & \boldsymbol{\phi}_{ijk} C^\top e_k \\ \star & \boldsymbol{\phi}_{ijk} \end{bmatrix} \succcurlyeq 0, \tag{45}$$

$$\begin{bmatrix} \boldsymbol{r}_{ijk} & \mathbf{0} & \mathbf{0} & \mathcal{G}_{ij}^\top(\mathbf{N}) & \mathbf{0} \\ \star & \mathbf{\Lambda}_{ijk} & \mathbf{0} & \mathcal{F}_{ij}^\top(\mathbf{N}) & \mathbf{0} \\ \star & \star & H_w^\top \mathbf{\Gamma}_{ijk} H_w & I_n & \mathbf{0} \\ \star & \star & \star & \mathbf{V}_{ijk} + \mathbf{V}_{ijk}^\top & \mathbf{V}_{ijk}^\top \\ \star & \star & \star & \star & \boldsymbol{X}_{ijk} \end{bmatrix} \succcurlyeq 0, \tag{46}$$

for $k \in \mathbb{N}_1^{n_c}$, $i \in \mathbb{N}_1^{2v_\theta}$ and $j \in \mathbb{N}_1^{v_p}$, where $\mathcal{G}_{ij}(\mathbf{N}), \mathcal{F}_{ij}(\mathbf{N})$ defined as in (42), and

$$\boldsymbol{r}_{ijk} := \boldsymbol{\phi}_{ijk} - {w^{\max}}^\top \mathbf{\Lambda}_{ijk} w^{\max} - \mathbf{1}_{n_w}^\top \mathbf{\Gamma}_{ijk} \mathbf{1}_{n_w}, \tag{47}$$

then, the controller with state-feedback gains characterized by $\mathbf{K} = U_p \mathbf{G}$, with $\mathbf{G}$ computed as $\mathbf{G}^l = \mathbf{N}^l \mathbf{W}^{-1}$ for $l \in \mathbb{N}_1^s$, renders the set $\mathcal{S}$ in (24) robust invariant.

Proof: The equality constraint (44) stems from the controller parameterization constraints given in (33). The remaining LMI conditions in (45)-(46) are proved as follows.

From the set definition (25), the invariance condition in (36) can be written as, for all $k \in \mathbb{N}_1^{n_c}$, $i \in \mathbb{N}_1^{2v_\theta}$ $j \in \mathbb{N}_1^{v_p}$,

$$1 - (e_k^\top C\theta^{i,j+})^2 \geq 0,\ \forall w \in \mathcal{W},\ \forall \overrightarrow{W_p} \in \mathcal{W}_p, \quad (48)$$

where e_k is the k-th column vector of the identity matrix I_{n_c}. By substituting the dynamics (43) in (48) we get,

$$1-(e_k^\top C\mathbf{W}^{-1}\mathbf{V}_{ijk}\xi_{ijk})^2 \geq 0,\ \forall w \in \mathcal{W}, \forall \overrightarrow{W_p} \in \mathcal{W}_p. \quad (49)$$

Following the S-procedure [21], we multiply (49) by a scalar variable $\phi_{ijk} > 0$ and lower bound its left hand side by a term which is non-negative for all $\left(w, \overrightarrow{W_p}\right) \in (\mathcal{W}, \mathcal{W}_p)$ as:

$$\begin{aligned}
&\phi_{ijk}(1 - (e_k^\top C\mathbf{W}^{-1}\mathbf{V}_{ijk}\xi_{ijk})^2) \geq \\
&\quad 2\xi_{ijk}^\top \underbrace{\left(\mathcal{G}_{ij}(\mathbf{N}) - \mathcal{F}_{ij}(\mathbf{N})\overrightarrow{W_p} + w - \mathbf{V}_{ijk}\xi_{ijk}\right)}_{0} \\
&\qquad + \underbrace{(w^{\max} + \overrightarrow{W_p})^\top \mathbf{\Lambda}_{ijk}(w^{\max} - \overrightarrow{W_p})}_{\geq 0} \\
&\qquad\quad + \underbrace{(\mathbf{1} + H_w w)^\top \mathbf{\Gamma}_{ijk}(\mathbf{1} - H_w w)}_{\geq 0}, \quad (50)
\end{aligned}$$

where we have employed Lemma 2 to express the symmetric polytopic sets $\mathcal{W}, \mathcal{W}_p$ as quadratic constraints, with $\mathbf{\Lambda}_{ijk} \in \mathbb{D}_+^{n(m+ns)}, \mathbf{\Gamma}_{ijk} \in \mathbb{D}_+^{n_w}$, as matrix decision variables.

A sufficient condition for invariance is obtained by re-arranging (50) into the following quadratic form:

$$\varkappa^\top \mathcal{P}_{ijk}(\mathbf{W}, \mathbf{N}, \mathbf{\Lambda}_{ijk}, \mathbf{\Gamma}_{ijk}, \phi_{ijk}, \mathbf{V}_{ijk})\varkappa \succcurlyeq 0,\ \forall \varkappa, \quad (51)$$

where $\varkappa^\top := \begin{bmatrix} 1 & -\overrightarrow{W_p}^\top & w^\top & -\xi_{ijk}^\top \end{bmatrix}$ and $\mathcal{P}_{ijk}$ is a symmetric matrix. The invariance condition is satisfied if $\mathcal{P}_{ijk} \succcurlyeq 0$, *i.e.*,

$$\begin{bmatrix}
\boldsymbol{r}_{ijk} & \mathbf{0} & \mathbf{0} & \mathcal{G}_{ij}^\top(\mathbf{N}) \\
\star & \mathbf{\Lambda}_{ijk} & \mathbf{0} & \mathcal{F}_{ij}^\top(\mathbf{N}) \\
\star & \star & H_w^\top \mathbf{\Gamma}_{ijk} H_w & I_n \\
\star & \star & \star & \mathbf{V}_{ijk}+\mathbf{V}_{ijk}^\top - \mathbf{V}_{ijk}^\top \mathcal{L}_{ijk}\mathbf{V}_{ijk}
\end{bmatrix} \succcurlyeq 0, \quad (52)$$

where $\mathcal{L}_{ijk} \triangleq \phi_{ijk}\mathbf{W}^{-\top}C^\top e_k e_k^\top C\mathbf{W}^{-1}$ and $\boldsymbol{r}_{ijk}$, is as defined in (47).

To resolve the non-linearity in the block $(4,4)$ of (52), we introduce a new matrix variable $\boldsymbol{X}_{ijk} = \boldsymbol{X}_{ijk}^\top \succ 0$ such that

$$\boldsymbol{X}_{ijk}^{-1} - \mathcal{L}_{ijk} \succ 0 \Leftrightarrow \boldsymbol{X}_{ijk}^{-1} - \phi_{ijk}\mathbf{W}^{-\top}C^\top e_k e_k^\top C\mathbf{W}^{-1} \succ 0. \quad (53)$$

By applying Schur complement to (53) we have,

$$\begin{bmatrix}
\boldsymbol{X}_{ijk}^{-1} & \phi_{ijk}\mathbf{W}^{-\top}C^\top e_k \\
\phi_{ijk}e_k^\top C\mathbf{W}^{-1} & \phi_{ijk}
\end{bmatrix} \succ 0, \quad (54)$$

which using the congruence transformation matrix $\text{diag}\{\mathbf{W}, I_n\}$, can be rewritten as

$$\begin{bmatrix}
\mathbf{W}^\top \boldsymbol{X}_{ijk}^{-1}\mathbf{W} & \phi_{ijk}C^\top e_k \\
\phi_{ijk}e_k^\top C & \phi_{ijk}
\end{bmatrix} \succ 0. \quad (55)$$

The nonlinearity $\mathbf{W}^\top \boldsymbol{X}_{ijk}^{-1}\mathbf{W}$ is resolved as follows,

$$\begin{aligned}
&\mathbf{W}^\top \boldsymbol{X}_{ijk}^{-1}\mathbf{W} = (\mathbf{W} - \boldsymbol{X}_{ijk})^\top \boldsymbol{X}_{ijk}^{-1}(\mathbf{W} - \boldsymbol{X}_{ijk}) + \mathbf{W} + \mathbf{W}^\top - \boldsymbol{X}_{ijk} \\
&\succcurlyeq \mathbf{W} + \mathbf{W}^\top - \boldsymbol{X}_{ijk} \quad (56)
\end{aligned}$$

Thus, replacing $\mathbf{W}^\top \boldsymbol{X}_{ijk}^{-1}\mathbf{W}$ in (55) with $\mathbf{W} + \mathbf{W}^\top - \boldsymbol{X}_{ijk}$, leads to a sufficient LMI condition for (55) as in (45), proving the first LMI condition stated in Theorem 4.

From (53), the condition (52) can be rewritten as

$$\begin{bmatrix}
\boldsymbol{r}_{ijk} & \mathbf{0} & \mathbf{0} & \mathcal{G}_{ij}^\top(\mathbf{N}) \\
\star & \mathbf{\Lambda}_{ijk} & \mathbf{0} & \mathcal{F}_{ij}^\top(\mathbf{N}) \\
\star & \star & H_w^\top \mathbf{\Gamma}_{ijk} H_w & I_n \\
\star & \star & \star & \mathbf{V}_{ijk}+\mathbf{V}_{ijk}^\top - \mathbf{V}_{ijk}^\top \boldsymbol{X}_{ijk}\mathbf{V}_{ijk}
\end{bmatrix} \succcurlyeq 0,$$

which followed by Schur complement gives the second LMI condition (46) stated in Theorem 4. ■

A. Volume maximization of the RCI set

As discussed in [11], [15], for a given C, the volume of the invariant set $\mathcal{S}$ in (8) is proportional to $|\det(\mathbf{W})|$. As in [15], to maximize the volume of the RCI set, we formulate an iterative determinant maximization problem while forcing it to grow between successive iterations. At the $q+1$-th iteration (with $q \in \mathbb{N}$), we impose the following constraint:

$$\mathbf{W}^\top W^q + (W^q)^\top \mathbf{W} - (W^q)^\top W^q \succcurlyeq \mathbf{W}_{\text{obj}} \succ 0, \quad (57)$$

where W^q and X_{ij}^q are the values of the variables $\mathbf{W}$, $\boldsymbol{X}_{ij}$ at the q-th iteration, and $\mathbf{W}_{\text{obj}} = \mathbf{W}_{\text{obj}}^\top \in \mathbb{R}^{n\times n}$ is the matrix to be optimized at the current iteration. This condition imposes a growth in the determinant *i.e.*, $|\det(W^{q+1})| \geq |\det(W^q)|$ and, hence, of the RCI volume at each iteration. By defining $Z_{ijk}^q := (X_{ijk}^q)^{-1}W^q$, the invariance condition in (45) can then be rewritten as [15]:

$$\begin{bmatrix}
\mathbf{W}^\top Z_{ijk}^q + (Z_{ijk}^q)^\top \mathbf{W} - (Z_{ijk}^q)^\top \boldsymbol{X}_{ijk} Z_{ijk}^q & \phi_{ijk}C^\top e_k \\
\phi_{ijk}e_k^\top C & \phi_{ijk}
\end{bmatrix} \succcurlyeq 0. \quad (58)$$

Accordingly, the problem to be solved at the $q+1$-th iteration is summarized as follows:

$$\begin{aligned}
\max_{\boldsymbol{Z}_{\text{SDP}}} \quad & \log\det(\mathbf{W}_{\text{obj}}) \\
\text{s.t.} \quad & (57), && \texttt{(successive growth)} \\
& (27), (32), && \texttt{(state/input constraints)} \\
& (44), && \texttt{(consistency constraint)} \\
& (45),\ (46), && \texttt{(invariance LMIs)}
\end{aligned} \quad (59)$$

where $\boldsymbol{Z}_{\text{SDP}} \triangleq (\mathbf{W}, \mathbf{N}, \boldsymbol{X}_{ijk}, \mathbf{V}_{ijk}, \phi_{ijk}, \mathbf{\Lambda}_{ijk}, \mathbf{\Gamma}_{ijk}, \mathbf{W}_{\text{obj}})$, for $k \in \mathbb{I}_1^{n_c}$, $i \in \mathbb{I}_1^{2v_\theta}$, $j \in \mathbb{I}_1^{v_p}$.

B. Computational complexity

The SDP program in (59) features $2v_\theta n_x$ and $2n_u v_\theta v_p$ scalar inequalities associated with the state and input constraints, respectively; s equality constraints induced by the controller parameterization constraint (33); $2n_c v_\theta v_p$ LMI constrains for set invariance (45)-(46). The number of rows in LMI (45) are $n+1$ while those in (46) are $1+n(m+ns)+3n$. The total number of scalar optimization variables defining the constraints and objective is $3n^2+(mn+n^2s)(s+1)+n_w+1$, which scales quadratically with the state dimension n, but, it is *independent* of data-length T.

VII. Numerical example

We demonstrate the effectiveness of the proposed approach via a numerical example. All algorithms have been implemented on an i7-1.40 GHz Intel core processor with 16 GB RAM running `MATLAB R2023a`, utilizing `MOSEK` [18] to solve the SDP programs.

Let us consider as the data-generating system the parameter-varying double integrator,

$$x_{k+1} = \begin{bmatrix} 1+\zeta_k & 1+\zeta_k \\ 0 & 1+\zeta_k \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k + w_k, \tag{60}$$

where $|\zeta_k| \leq 0.2$. The following state-input constraints are imposed on the system,

$$\mathcal{X} := \{x : \|x\|_\infty <= 5\}, \quad \mathcal{U} := \{u : |u| \leq 3\}.$$

The disturbance is assumed to be bounded in a set $\mathcal{W} = \{w : |w| \leq 0.05\}$. The system can be expressed in the LPV-A affine form in (1) with

$$A^1 = \begin{bmatrix} 1.2 & 1.2 \\ 0 & 1.2 \end{bmatrix}, A^2 = \begin{bmatrix} 0.8 & 0.8 \\ 0 & 0.8 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

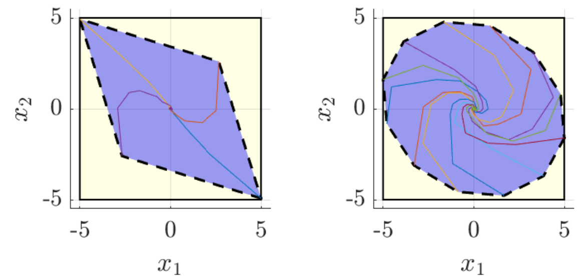
using $p_{k,1} = 2(0.2 + \zeta_k)$, $p_{k,2} = 2(0.2 - \zeta_k)$, with scheduling parameter belongs to the set $\mathcal{P} = \{p \in \mathbb{R}^2 : p \in [0,1], p_1 + p_2 = 1\} = \mathsf{Conv}([\begin{smallmatrix}1\\0\end{smallmatrix}],[\begin{smallmatrix}0\\1\end{smallmatrix}])$. The system matrices $\{A^1, A^2, B\}$ are *unknown* and are not used for the computation of the RCI set or the controller. A single state-input-scheduling trajectory of $T = 20$ samples is gathered by exciting the system (60) with input signal uniformly sampled in the interval $[-3, 3]$. The measured states are corrupted by a disturbance signal uniformly distributed in the interval $[-0.05, 0.05]$. The collected data satisfies the rank condition in Assumption 1, *i.e*, $\text{rank}(\Phi) = 5$.

The cross-covariance bounds given in (37) for this example are considered as: $c_j = \{0.0061\mathbf{1}_n, 0.05921\mathbf{1}_n, 0.01051\mathbf{1}_n, 0.04031\mathbf{1}_n, 0.00341\mathbf{1}_n\}$, such that disturbance satisfies the $w_k \in \mathcal{W}, \forall k \in \mathbb{N}_1^{20}$. If these bounds are assumed to be not known a priori, we compute the worst-case approximation by solving the LP (40). then, the computed bounds are given as $w^{\max} = \text{vec}\{0.08871\mathbf{1}_n, 0.12931\mathbf{1}_n, 0.02591\mathbf{1}_n, 0.13831\mathbf{1}_n, 0.05451\mathbf{1}_n\}$.

We compute the RCI sets for the following two cases: 1) RCI set denoted by $\mathcal{S}_0$: computed by assuming known cross-covariance bounds; and 2) RCI set denoted by $\mathcal{S}_1$: computed with unknown cross-covariance bounds, and estimated with the available data solving LP in (40).

For analyzing the effect of choice of C defining the representational complexity of the RCI set, in our tests, we choose two different C matrices with representational complexities $n_c = 2, 6$. Each row of C is chosen as follows, see [11, Remark 1]: $e_i^\top C = \left[\cos\left(\frac{\pi(i-1)}{n_c}\right), \ \sin\left(\frac{\pi(i-1)}{n_c}\right)\right], i \in \mathbb{I}_1^{n_c}$. The RCI set and the associated LPV state-feedback gain matrices are computed by solving (59) for 5 iterations.

In Fig. 1, we plot the obtained RCI sets $\mathcal{S}_0$ for $n_c = 2, 6$. As seen from the figure, RCI set with $n_c = 6$ is larger w.r.t. to the one with $n_c = 2$. The choice of C and its representational complexity n_c, affects the volume of the RCI set, and it can be used to achieve a trade-off between complexity *vs* size of the set. The corresponding invariant set matrix $\mathbf{W}$ and feedback control gains $\{\mathbf{K}^1, \mathbf{K}^2\}$ are reported below:



(a) $n_c = 2$ (b) $n_c = 6$

Fig. 1: RCI sets $\mathcal{S}_0$ (violet) with closed-loop simulated trajectories and state constraints set $\mathcal{X}$ (yellow).

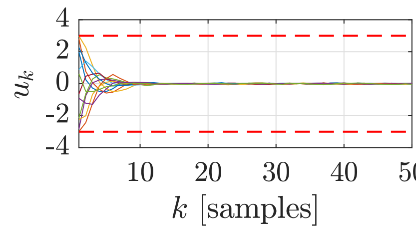


Fig. 2: Control input $u = \mathbf{K}(p)x$ trajectories for the computed state-feedback gain. Input constraints (dashed-red).

$$n_c = 2: \quad \begin{bmatrix} \mathbf{W} \\ \hline \mathbf{K}^1 \\ \hline \mathbf{K}^2 \end{bmatrix} = \begin{bmatrix} 3.8387 & -1.1613 \\ -1.1613 & 3.7635 \\ \hline -0.3015 & -0.6685 \\ \hline -0.2863 & -0.8583 \end{bmatrix},$$

$$n_c = 6: \quad \begin{bmatrix} \mathbf{W} \\ \hline \mathbf{K}^1 \\ \hline \mathbf{K}^2 \end{bmatrix} = \begin{bmatrix} 4.8976 & -0.3940 \\ -0.3940 & 4.6743 \\ \hline -0.2977 & -0.5804 \\ \hline -0.3536 & -0.5882 \end{bmatrix}$$

Fig. 1 also depicts closed-loop state trajectories starting from each vertex of the RCI set and corresponding control input trajectories are shown in Fig. 2, which shows that the input constraints are satisfied during the closed-loop simulation. The state trajectories are obtained by simulating the true system (60) in closed-loop with the obtained state-feedback controller $u = \mathbf{K}(p)x$ with each vertex of the RCI set as the initial condition. Note that for each closed-loop simulation, we generate a different realization of the scheduling signal taking values in the given bound $p \in [0, 1]$, as well as a different realization of the disturbance signal

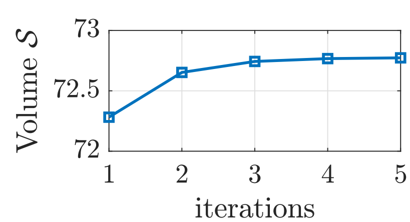


Fig. 3: Volume of $\mathcal{S}_0$ *vs* iterations of SDP (59)

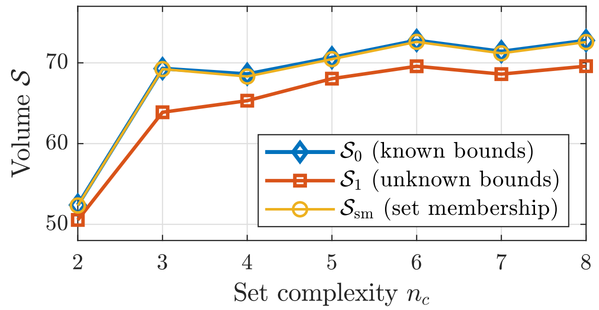


Fig. 4: Volume *vs* representational complexity n_c for RCI sets, computed with known cross-covariance bounds $w^{\max}$ ($\mathcal{S}_0$), estimated $w^{\max}$ ($\mathcal{S}_1$), set-membership approach ($\mathcal{S}_{\rm sm}$) [15].

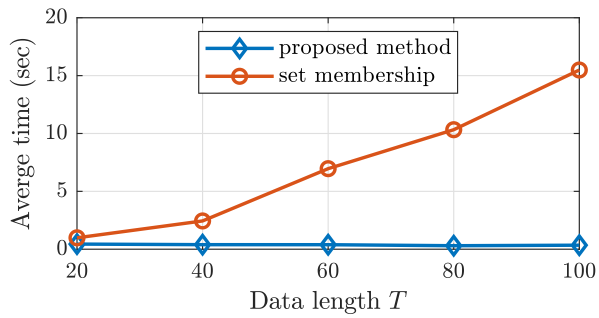


Fig. 5: Average computation time per iteration *vs* length of the data set T.

$w \in \mathcal{W}$ acting on the system at each time instance. Hence, in the presence of a bounded but unknown disturbance, the result shows that the approach guarantees robust invariance for all possible scheduling signals in a given set, while always respecting the state and input constraints. The volume of the RCI sets w.r.t. iterations of SDP in (59) is shown in Fig. 3, it can be seen that no significant volume increase is observed after 3 iterations.

In order to asses the size of the obtained sets, we compare the volumes of the sets $\mathcal{S}_0, \mathcal{S}_1$ (obtained respectively by known and estimated cross-covariance bounds), with RCI sets $\mathcal{S}_{\rm sm}$ computed by the set-membership approach [15], as depicted in Fig 4 for varying representational complexity n_c. We observe that the proposed algorithm with data-based closed-loop parameterization can produce RCI set that are comparable in size to those given by [15]. The sets $\mathcal{S}_1$ computed with estimated bounds solving (40), have slightly lower volumes. Estimated cross-covariance bounds leads to an expected conservatism due to the over-approaximation and hence, slightly smaller RCI sets.

The main advantage of the proposed approach over [15] can be seen in terms of the computation time as shown in Fig 5. As the number of data samples grows, the average computation time to solve one iteration of SDP in (59) does not increase, while it increases linearly with T when considering one SDP iteration of Algorithm 1 in [15].

VIII. CONCLUSION

In this work, we proposed a direct approach based on data-driven closed-loop parameterization, to compute RCI sets and controllers for LPV systems, as an alternative to the direct approach of [15]. Our results show that the proposed algorithm generates RCI sets that are of comparable size to the ones obtained with the set-membership framework in [15], but it outperforms [15] from a computational perspective. In particular, we have shown that assuming cross-covariance noise bounds, the proposed method can be computationally efficient in that the number of decision variables do not grow with size of the dataset.

Future work will involve developing an algorithm incorporating performance guarantees within an RCI set and extensions for switched dynamical systems.

IX. APPENDIX: PROOF OF LEMMA 3

Proof: $(i) \Rightarrow (ii)$: For each vertex $\theta^i,\ i \in \mathbb{N}_1^{2v_\theta}$, and $p^j,\ j \in \mathbb{N}_1^{v_p}$, it holds that $\theta^i \in \Theta, p^j \in \mathcal{P}$, thus, $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$: Any given $\theta \in \Theta$ and $p \in \mathcal{P}$ can be expressed as a convex combination of the vertices of the respective sets, *i.e.*,

$$\theta = \sum_{i=1}^{2v_\theta} \alpha_i \theta^i, \quad \sum_{i=1}^{2v_\theta} \alpha_i = 1, \quad \alpha_i \geq 0$$
$$p = \sum_{j=1}^{v_p} \beta_j p^j, \quad \sum_{j=1}^{v_p} \beta_j = 1, \quad \beta_j \geq 0$$

Then, dynamics in (i) can be written as,

$$\theta^+ = \mathbf{W}^{-1}(X_p^+ - W_p)\mathbf{N}\left(\sum_{j=1}^{v_p}\beta_j p^j \otimes \sum_{i=1}^{2v_\theta}\alpha_i\theta^i\right) + \mathbf{W}^{-1}w \tag{61}$$

$$= \sum_{j=1}^{v_p}\beta_j\sum_{i=1}^{2v_\theta}\alpha_i \underbrace{\left(\mathbf{W}^{-1}(X_p^+ - W_p)\mathbf{N}(p^j \otimes \theta^i) + \mathbf{W}^{-1}w\right)}_{\theta^{i,j+}\in\Theta} \tag{62}$$

$$= \sum_{j=1}^{v_p}\beta_j \underbrace{\sum_{i=1}^{2v_\theta}\alpha_i\theta^{i,j+}}_{\theta^{j+}\in\Theta} = \sum_{j=1}^{v_p}\beta_j\theta^{j+} \in \Theta, \tag{63}$$

where (62) follows from the distributive property of the Kronecker product. From (ii), we have $\theta^{i,j+} \in \Theta$ (see (36)). As θ^{j+} in (63) is a convex combination of $\theta^{i,j+}$ and as the set Θ is convex, it follows that $\theta^{j+} \in \Theta$. Finally, as θ^+ is obtained as a convex combination of $\theta^{j+} \in \Theta$ and as the set Θ is convex, it follows that $\theta^+ \in \Theta$, thus proving $(ii) \Rightarrow (i)$. ■

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