Regularized Moving-Horizon PWA Regression for LPV System Identification

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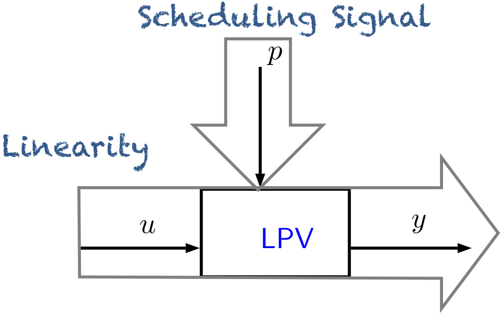
Linear Parameter-Varying (LPV) Concept



- LPV paradigm: Linear dynamic relation between input and output
- Unlike Linear Time-Invariant (LTI), the relation changes over time according to a measurable time-varying 'scheduling signal' p.

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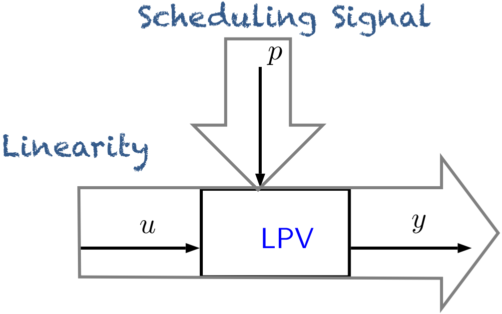

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LPV Model Identification

LPV-Input Output model:

$$y(k) = a_0(p(k)) + \sum_{j=1}^{n_a} a_j(p(k)) y(k-j) + \sum_{j=1}^{n_b} a_{j+n_a}(p(k)) u(k-j),$$

- $y(k), u(k)$ measured outputs and inputs at time k
- $p(k)$ measured scheduling variable, values in a set $\mathcal{P} \subseteq \mathbb{R}^{n_p}$

Objective

Given the dataset of N observations: $\{y(k), u(k), p(k)\}_{k=1}^{N}$,

- Estimate the p-dependent coefficient functions $a_j(p(k))$.

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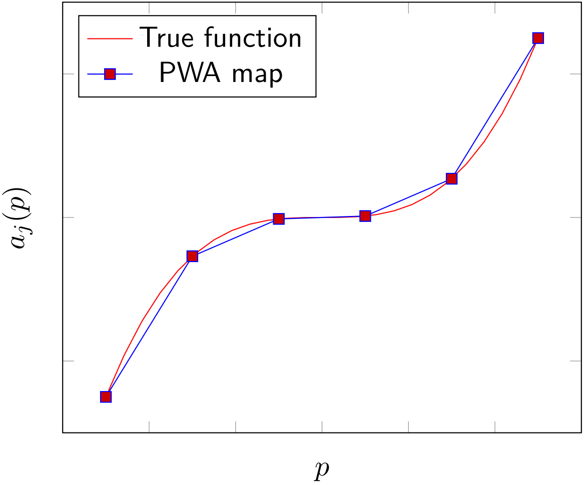
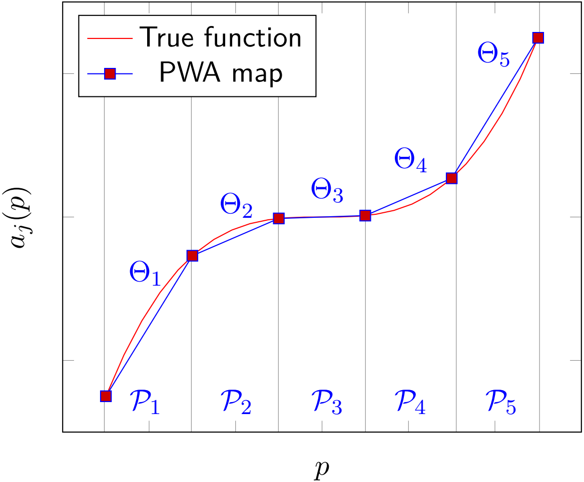
LPV Identification as PWA regression



- Proposed idea: Approximate non-linear functions $a_j(p(k))$ with PieceWise Affine (PWA) maps

LPV Identification as PWA regression



$$a_j(p(k)) = \begin{cases} \Theta_{1,j}\ p & \text{if } p \in \mathcal{P}_1, \\ \vdots & \\ \Theta_{s,j}\ p & \text{if } p \in \mathcal{P}_s, \end{cases}$$

LPV Identification as PWA regression

- LPV-Input Output model

$$y(k) = a_0(p(k)) + \sum_{j=1}^{n_a} a_j(p(k)) y(k-j) + \sum_{j=1}^{n_b} a_{j+n_a}(p(k)) u(k-j),$$

can be written in the following PWA-LPV form:

$$y(k) = \begin{cases} \Theta_1 \; x(k) & \text{if } p(k) \in \mathcal{P}_1, \\ \vdots & \\ \Theta_s \; x(k) & \text{if } p(k) \in \mathcal{P}_s. \end{cases}$$

- $x(k)$ the regressor vector

$$x(k) = \begin{bmatrix} 1 & y'(k-1) & \cdots & y'(k-n_a) & u'(k-1) & \cdots & u'(k-n_b) \end{bmatrix}' \otimes \begin{bmatrix} 1 \\ p(k) \end{bmatrix}.$$

LPV Identification as PWA regression

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Estimation of the PWA-LPV model consists of:

1. selecting the number of modes s
2. estimating the model parameter matrices Θ_i,
3. computing the polyhedra $\{\mathcal{P}_i\}_{i=1}^{s}$, defining partition of scheduling variable space

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Estimation of the PWA-LPV model consists of:

1. selecting the number of modes s (via cross-calibration)
2. estimating the model parameter matrices Θ_i,
3. computing the polyhedra $\{\mathcal{P}_i\}_{i=1}^{s}$, defining partition of scheduling variable space

PWA regression algorithm

The developed algorithm for PWA regression consists of the following two stages:

Stage S1.
- *estimation* of the model parameters Θ_i
- simultaneous *clustering* of the scheduling variables $\{p(k)\}_{k=1}^{N}$
- using regularized moving horizon regression algorithm

Stage S2.
- *Computation of polyhedral partitions* of the scheduling variable space $\mathcal{P}$
- using computationally efficient multi-category linear separation methods.

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Stage S1: Regularized moving horizon identification algorithm

- At each time sample k, a moving-horizon window of length T containing training data samples $\{x(k), y(k), p(k)\}$ from time $k - T + 1$ to time k is considered.
- A *mixed-integer programming problem* is solved to simultaneously estimate:
  1. the model parameters Θ_i
  2. the active mode $\sigma(k) \in \{1, \dots, s\}$ which determines the polyhedral partition $\mathcal{P}_i$ in which $p(k)$ belongs to at time k, i.e., $\boxed{\sigma(k) = i^* \Rightarrow p(k) \in \mathcal{P}_{i^*}}$
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At time k, we solve,

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$$\text{s.t.}\ \ \delta_i(k-t) \in \{0,1\},\ \sum_{i=1}^{s} \delta_i(k-t) = 1,\ t = 0, \ldots, T-1.$$

- Fitting error term optimized over T horizon samples
- $\delta_i(k)$ indicates active mode at time k, i.e., $\boxed{\delta_i(k) = 1 \Rightarrow \sigma(k) = i \Rightarrow p(k) \in \mathcal{P}_i}$
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$$+ \boxed{\sum_{t=1}^{k-T} \left\| y(t) - \Theta_{\sigma(t)} x(t) \right\|^2} \quad \text{regularization on } \Theta_i$$

$$\text{s.t.} \quad \delta_i(k-t) \in \{0,1\}, \ \sum_{i=1}^{s} \delta_i(k-t) = 1, \quad t = 0, \ldots, T-1.$$

- takes into account the time history, outside the considered time window i.e. from time 1 to time $k-T$
- Fixed estimates $\{\sigma(t)\}_{t=1}^{k-T}$ are used, to refine the estimates of the model parameters Θ

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$$\text{s.t.}\ \ \delta_i(k-t) \in \{0,1\},\ \sum_{i=1}^{s} \delta_i(k-t) = 1,\ t = 0, \ldots, T-1.$$

- Clustering: penalizes the error between $p(k)$ and centroids $\{c_i\}_{i=1}^{s}$ of clusters $\mathcal{P}_i$

Stage S1: Regularized moving horizon identification algorithm

$$
\begin{aligned}
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& + \boxed{\sum_{t=0}^{T-1} \sum_{i=1}^{s} \|(p(k-t) - c_i)\delta_i(k-t)\|^2} \quad \text{centroid distance} \\
\text{s.t.} \quad & \delta_i(k-t) \in \{0,1\},\ \sum_{i=1}^{s} \delta_i(k-t) = 1,\ t = 0, \ldots, T-1.
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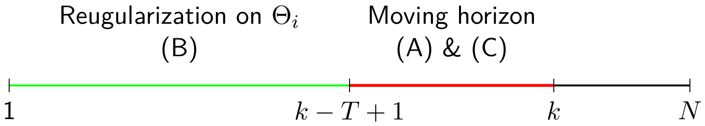

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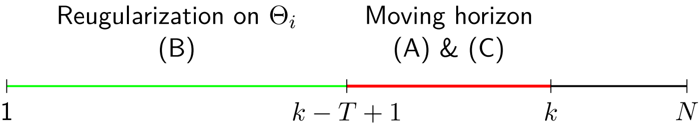

Mixed-Integer Problem

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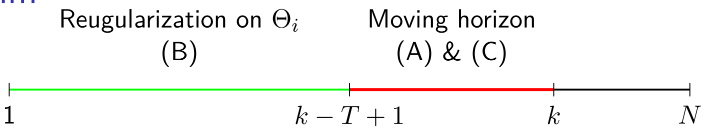

Mixed-Integer Problem

- Only the active mode $\sigma(k)$ at time k is kept
- the T-length time window is shifted forward to estimate the next mode $\sigma(k+1)$ solving the Mixed Integer Quadratic Programming (MIQP) problem.

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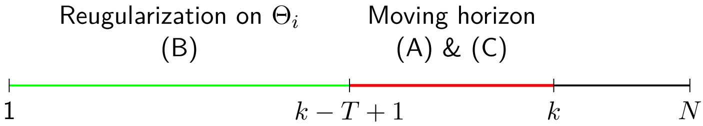

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Iterative Refinement

- Initially for $k \ll N$, the clustering may be inaccurate
- It affects the evaluation of mode $\sigma(k)$
- As the regularization cost on Θ_i also depends on the estimated sequence $\{\sigma(t)\}_{t=1}^{k}$
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- We run the algorithm in Stage S1 multiple times (n_q) for iterative refinement by working in a batch mode

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- this is done by further adding a regularization term

$$\sum_{q=1}^{n_q-1} \lambda^{n_q-q-1} \sum_{t=1}^{N-T} \left\| y(t) - \Theta_{\sigma(t,q)} x(t) \right\|^2,$$

with $\sigma(t,q)$: estimate of the active mode at time t obtained at the q-th run of moving horizon algorithm

- A forgetting factor $\lambda \in \mathbb{R} : 0 < \lambda \leq 1$ is also included to exponentially downweight the estimates obtained at past runs
- For both the Regularization terms, a recursive update of the objective function is proposed to avoid the need to store the time history of the observations

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Summary of Stage S1

- The model parameters Θ_i of PWA function have been estimated.
- Clustering of $\{p(k)\}_{k=1}^{N}$ into s clusters based on estimated mode sequence $\sigma(k)$ is obtained.
- Each cluster corresponds to a polyhedral partition $\mathcal{P}_i$.

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Stage S2: Linear multicategory discrimination

Computation of a polyhedral partition $\mathcal{P}_i$ of the scheduling variable space $\mathcal{P}$ is done using the PWA separator function ϕ is defined as

$$\phi(p) = \max_{i=1,\ldots,s} \left(p'\omega^i - \gamma^i\right),$$

the polyhedra $\{\mathcal{P}_i\}_{i=1}^s$ are defined as

$$\mathcal{P}_i = \left\{p \in \mathbb{R}^{n_p} : [p' \quad -1]\left[\begin{smallmatrix}\omega^i - \omega^j \\ \gamma^i - \gamma^j\end{smallmatrix}\right] \geq 1,\ j = 1, \ldots, s,\ j \neq i\right\}.$$

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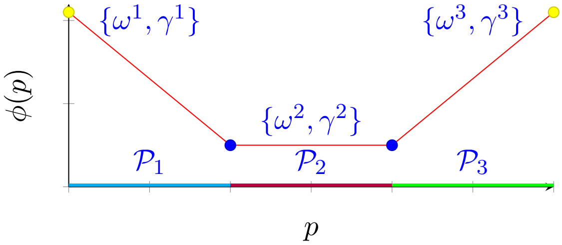

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Stage S2: Linear multicategory discrimination

The parameters $\{\omega^i, \gamma^i\}_{i=1}^s$ are calculated by solving the optimization problem which is convex, (Breschi, Piga and Bemporad, 2016)

$$\min_{\omega^i,\gamma^i} \frac{\kappa}{2} \sum_{i=1}^{s} \left(\|\omega^i\|_2^2 + (\gamma^i)^2 \right) +$$

$$\sum_{i=1}^{s} \sum_{\substack{j=1 \\ j \neq i}}^{s} \frac{1}{m_i} \left\| \left(\begin{bmatrix} M_i & -\mathbf{1}_{m_i} \end{bmatrix} \begin{bmatrix} \omega^j - \omega^i \\ \gamma^j - \gamma^i \end{bmatrix} + \mathbf{1}_{m_i} \right)_+ \right\|_2^2$$

Numerical Examples

Identification of SISO LPV-ARX system

Single-Input Single-Output (SISO) LPV-ARX system:

$$y(k) = a_1^{\mathrm{o}}(p(k))y(k-1)+a_2^{\mathrm{o}}(p(k))y(k-2)+b_1^{\mathrm{o}}(p(k))u(k-1)+e(k),$$

$$a_1^{\mathrm{o}}(p(k)) = \begin{cases} -0.5, & \text{if } p(k) > 0.5 \\ -p(k), & \text{if } -0.5 \leq p(k) \leq 0.5 \\ 0.5, & \text{if } p(k) < -0.5 \end{cases}$$

$$a_2^{\mathrm{o}}(p(k)) = p^3(k), \quad b_1^{\mathrm{o}}(p(k)) = \sin(\pi p(k)).$$

with $N = 6000$ training data, SNR on the output channel: 20 dB.

- A PWA model with $s = 6$ modes is considered.
- Stage S1 is run with prediction horizon $T = 6$.
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Identification of SISO LPV-ARX system

- The resultant MIQP: 36 binary and 72 continuous variables, 144 inequality and 6 equality constraints.
- The mean time to solve: GUROBI (commercial) 0.09 sec, GPAD-B&B [1] 0.13 sec
- GPAD-B&B is simple library-free solver, yet rendered comparable performance w.r.t GUROBI

Best Fit Rate

- BFR on noise-free validation dataset $N_{\mathrm{val}} = 2000$ is 0.95

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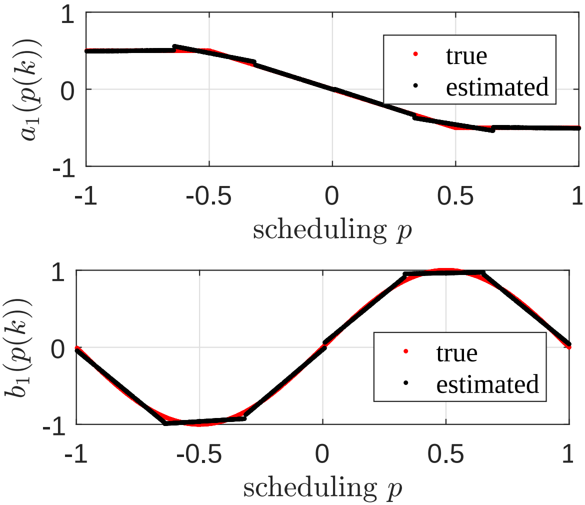
Estimated LPV coefficient functions



Identification of MIMO LPV-ARX system

Multi-Input Multi-Output (SISO) LPV-ARX system:

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with $N = 6000$ training data samples,
SNR on the output channels: $\mathrm{SNR}_1 = 23$ dB, $\mathrm{SNR}_2 = 23.5$ dB.

- A PWA model with $s = 10$ modes is considered.
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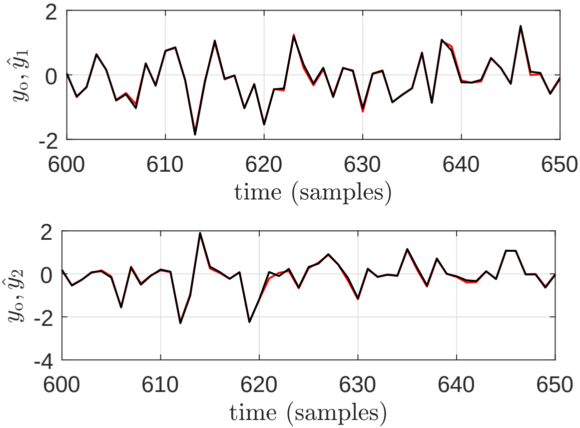
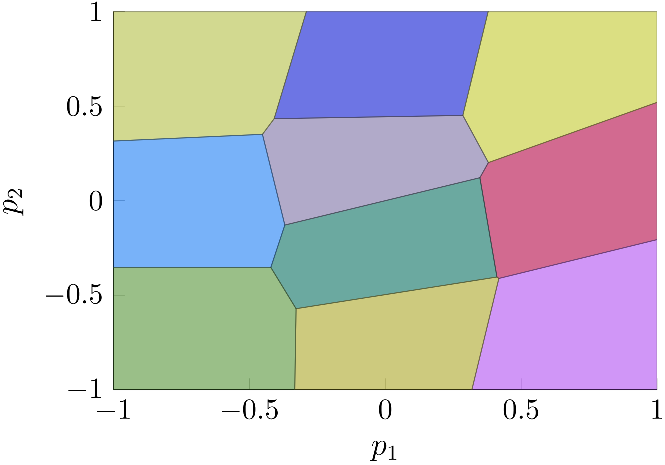


Best Fit Rates

runs(n_q)	BFR y_1	BFR y_2
1	0.8793	0.8108
2	0.8817	0.8146

Scheduling space partition



Conclusions

- LPV identification is recast as *PieceWise Affine* (PWA) regression problem.
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Acknowledgment

The authors would like to thank Valentina Breschi for her help with the multicategory discrimination algorithm.

Thank You